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Applications of the Dirac Form of the Maxwell Equations in Moving Dielectric Media

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Abstract Maxwell equations in a resting and nonrelativisticly moving medium can be rewritten in a form of the Dirac equation. In the paper the formal analogy between an electron in the electromagnetic field and a photon in the dielectric medium has been used to consider three effects: Fresnel's drag, mechanical Faraday effect (interpreted here as a procession of the photon spin) and Landau frequencies in a rotating medium. The third effect, up to my knowledge, is new. It predicts that only some discrete frequencies of light can propagate in a rotating medium.

Keywords Maxwell equations · Moving dielectric media · Dirac equation · Photon

1 Introduction

In 1907 L. Silberstein [1, 2] observed that it is possible to write Maxwell equations in a concise form if one uses a complex field

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}. \tag{1}$$

I. Bialynicki-Birula [3, 4] constructed the photon wave function in a coordinate representation as the positive frequency (energy) part of the following object

$$\psi = \begin{bmatrix} \mathbf{E}^{(+)} + i\mathbf{H}^{(+)} \\ \mathbf{E}^{(+)} - i\mathbf{H}^{(+)} \end{bmatrix}. \tag{2}$$

His approach was the starting point for my considerations presented in [5]. It was shown there that formally the propagation of a photon in a medium may be considered as a propagation of a massive particle in an external field described by scalar and vector potentials of the medium. The scalar potential refers to susceptibilities of the medium, and the vector

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potential to the motion of the medium. Both relativistic and nonrelativistic velocities of the medium were considered. The aim of the present paper is to develop that analogy and to present its usefulness by giving some examples.

The Gaussian system of units has been used in the paper. However, for convenience the coefficient 4π has been plugged into the definition of susceptibilities.

2 Dirac Equation for Photon

Assume that a source of light and an observer are at relative rest and the medium moves relative to them. For nonrelativistic velocities the Maxwell equations in the medium can be written in the form of the following Schrödinger equation [5]

$$E\psi_\omega = c(\mathbf{P}-\mathbf{A})\cdot\begin{bmatrix}\mathbf{S} & 0\\ 0 & -\mathbf{S}\end{bmatrix}\psi_\omega - \Omega_\omega\gamma\psi_\omega - \Gamma_\omega\eta\psi_\omega. \tag{3}$$

$E=\hbar\omega$ is the total energy of the photon. $\mathbf{A}=(\hbar\omega/c)\mathbf{a}$ is the vector potential of the medium, where $\mathbf{a}$ is the dimensionless vector potential $\mathbf{a}=(1-\varepsilon_{\omega 0}\mu_{\omega 0})\vec{\beta}$ (here $\vec{\beta}=\mathbf{u}/c$ and $\mathbf{u}$ is a velocity of the medium). $\mathbf{S}$ is the photon spin operator determined in Cartesian representation by the matrix components $(\mathbf{S}_i)_{kl}=-i\epsilon_{ikl}$, where ϵ_{ikl} is antisymmetric Levi-Civita symbol. γ and η are projective operators splitting the wave function into electric and magnetic parts $\gamma\psi_\omega+\eta\psi_\omega=\psi_\omega$.

$$\gamma\psi_\omega=\begin{bmatrix}\mathbf{E}_\omega\\ \mathbf{E}_\omega\end{bmatrix},\qquad \eta\psi_\omega=\begin{bmatrix}i\mathbf{H}_\omega\\ -i\mathbf{H}_\omega\end{bmatrix}, \tag{4}$$

$$\gamma=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\qquad \eta=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}. \tag{5}$$

The photon in a medium is regarded here as a particle with an "effective mass" m moving in a "classical potential field" U

$$mc^2=\frac{|\Omega_\omega-\Gamma_\omega|}{2},\qquad U=-\frac{\Omega_\omega+\Gamma_\omega}{2}. \tag{6}$$

Ω_ω and Γ_ω refer to susceptibilities of the medium in the following way (see (A.3) in Appendix)

$$\Omega_\omega=\hbar\omega\chi_{\omega_0},\qquad \Gamma_\omega=\hbar\omega\chi^m_{\omega_0}. \tag{7}$$

The subscript ω_0 refers to the frequency of the photon in the reference frame in which the medium is at rest. Due to the motion of the medium the frequency ω in the frame of the source is different from ω_0. Certainly, the exact relation between them depends on the kind of motion of the medium. Generally it may be written as some function

$$\omega_0=\omega_0(\omega). \tag{8}$$

For nonrelativistic velocities of the medium one may assume that the frequency ω_{obs} in the observer frame is the same as in the source frame, i.e. $\omega=\omega_{obs}$. Although the source and the observer are at relative rest the equality is not exactly true as it has been shown in Appendix. The distinction between the three frequencies ω, ω_0, ω_{obs} was not taken into account in my previous paper [5].

The Schrödinger equation (3) is strict in the case of a uniformly moving homogeneous medium. In other cases it will be assumed that the flow $\mathbf{u}$ and the potentials Ω, Γ vary only gradually, i.e., do not vary significantly over one optical wavelength and cycle. To represent a complete set of Maxwell equations, (3) must be supplemented by divergence condition $\mathbf{p}\cdot\psi_\omega = 0$, where $\mathbf{p} = \mathbf{P} - \mathbf{A}$ is the kinetic momentum, and $\mathbf{P} = -i\hbar\nabla$ is the canonical momentum of the photon. The divergence condition is strictly fulfilled for uniformly moving homogeneous medium. In other cases it may be treated as a good approximation.

One may invert (6) and replace the couplings Ω_ω, Γ_ω in (3) by the m and U. There are two cases

1. $(\Omega_\omega - \Gamma_\omega) < 0$, then $mc^2 = -\frac{\Omega_\omega - \Gamma_\omega}{2}$,
2. $(\Omega_\omega - \Gamma_\omega) > 0$, then $mc^2 = \frac{\Omega_\omega - \Gamma_\omega}{2}$

and appropriately two following equations

$$\left[E - U - c\vec{\alpha}\cdot(\mathbf{P} - \mathbf{A}) - mc^2\beta\right]\psi_\omega = 0, \tag{9}$$

$$\left[E - U - c\vec{\alpha}\cdot(\mathbf{P} - \mathbf{A}) + mc^2\beta\right]\psi_\omega = 0. \tag{10}$$

The matrices $\vec{\alpha}$ and β are defined in the following way

$$\vec{\alpha} = \begin{bmatrix} \mathbf{S} & 0 \\ 0 & -\mathbf{S} \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

Note that $\gamma - \eta = \beta$.

Write now

$$U = \hbar\omega\varphi, \quad \text{where “potential” } \varphi = -\frac{1}{2}(\chi_{\omega_0} + \chi^m_{\omega_0}). \tag{12}$$

Note that φ depends on the medium characteristics only. To emphasize more the similarity with the Dirac equation of electron one may introduce the notion of an “optical charge” as

$$e_{opt} = \hbar\omega. \tag{13}$$

The definition seems artificial or excessive because the quantity $\hbar\omega$ has been already defined as the total energy E of the photon. Remember however that in classical electrodynamics electric charge is simply a coupling parameter between the charged particle, e.g., electron and the electromagnetic field potentials [7]. The coupling of photon with the medium is determined by its frequency, therefore the frequency (energy) of the photon acts as electric charge in the case of electron. This in a way justifies the introduction of the term “optical charge”. Thus, (9) and (10) take on the forms

$$\left[E - e_{opt}\varphi - \vec{\alpha}\cdot(c\mathbf{P} - e_{opt}\mathbf{a}) - mc^2\beta\right]\psi_\omega = 0, \tag{14}$$

$$\left[(-E) - (-e_{opt})\varphi - \vec{\alpha}\cdot(-c\mathbf{P} - (-e_{opt})\mathbf{a}) - mc^2\beta\right]\psi_\omega = 0. \tag{15}$$

The equation (14) has exactly the same form as the well known Dirac equation. The difference is only in definitions of matrices $\vec{\alpha}$ and β, because the Pauli matrix vector $\vec{\sigma}$ has been replaced by matrix vector $\mathbf{S}$ (in spinor representation) and the dimension of the matrices is six.

If we consider energy E in (14) as positive, then the second equation (15) describes the photon with energy $-E$, momentum $-\mathbf{P}$, and “optical charge” $-e_{opt}$. But photons, even photons in medium, have no antiparticles! As a matter of fact, we can easily recast (15) into the form of (14). To this end one should change in (15) $\omega \to -\omega$ (which means the simultaneous changes $E \to -E$ and $e_{opt} \to -e_{opt}$). Note however, that this operation does not change the susceptibilities, because real susceptibilities are even functions of frequency [6]. Making next the complex conjugation of (15) and multiplying it from left by the matrix β one obtains

$$\left[E - e_{opt}\varphi - \vec{\alpha}\cdot(c\mathbf{P} - e_{opt}\mathbf{a}) - mc^2\beta\right]\tilde{\psi}^*_{-\omega} = 0, \tag{16}$$

where

$$\tilde{\psi}_\omega = \beta\psi_\omega = \begin{bmatrix}\mathbf{E}_\omega - i\mathbf{H}_\omega \\ \mathbf{E}_\omega + i\mathbf{H}_\omega\end{bmatrix}. \tag{17}$$

Taking into account

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) + i\mathbf{H}(\mathbf{r},t) \equiv \mathbf{F}(\mathbf{r},t) &= \int_{-\infty}^{\infty}\mathbf{F}_\omega(\mathbf{r})e^{-i\omega t}d\omega = \int_{-\infty}^{\infty}(\mathbf{E}_\omega + i\mathbf{H}_\omega)e^{-i\omega t}d\omega,\\ \mathbf{E}(\mathbf{r},t) - i\mathbf{H}(\mathbf{r},t) \equiv \tilde{\mathbf{F}}(\mathbf{r},t) &= \int_{-\infty}^{\infty}\tilde{\mathbf{F}}_\omega(\mathbf{r})e^{-i\omega t}d\omega = \int_{-\infty}^{\infty}(\mathbf{E}_\omega - i\mathbf{H}_\omega)e^{-i\omega t}d\omega\\ &= \mathbf{F}^*(\mathbf{r},t),\end{aligned}$$

we find $\mathbf{F}^*{}_{-\omega} = \tilde{\mathbf{F}}_\omega$, and immediately infer from it that $\tilde{\psi}^*_{-\omega} = \psi_\omega$. Thus (14) and (15) are equivalent.

For *relativistic* velocities of the medium, the form of the Schrödinger equation is more complicated ([5], equation (35)), because the motion of the medium produces anisotropy of the whole system which cannot be neglected. Therefore, it is possible to rearrange the Schrödinger equation to the form similar to the Dirac equation only in the special case when the light propagates in the same direction as the medium moves. Nevertheless, in order to consider propagation of light in *any* direction I will consider in this paper mainly nonrelativistic velocities of the medium.

3 Fresnel's Drag of Light

We may use the photon Dirac equation (14) to consider the propagation of the plane wave in the moving medium.

$$\psi_\omega = \psi_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{18}$$

where ψ_0 is a constant six-component spinor and $\mathbf{k}$ wave vector. One has to solve the following characteristic equation

$$\det\left[E - e_{opt}\varphi - \vec{\alpha}\cdot(c\hbar\mathbf{k} - e_{opt}\mathbf{a}) - mc^2\beta\right] = 0. \tag{19}$$

Thus

$$E = \sqrt{c^2(\hbar\mathbf{k} - e_{opt}\mathbf{a}/c)^2 + e_{opt}^2\frac{(\chi_{\omega 0} - \chi^m_{\omega 0})^2}{4}} - \frac{1}{2}e_{opt}\left(\chi_{\omega 0} + \chi^m_{\omega 0}\right). \tag{20}$$

The transition from the "quantum" to "classical" description consists in replacing E and e_{opt} with $\hbar\omega$. After some algebra from (20) one obtains

$$\frac{k^2}{\omega^2} - 2\,\frac{k}{\omega}\,\frac{\mathbf{u}\cdot\hat{\mathbf{k}}}{c^2}\left(1-\varepsilon_{\omega_0}\mu_{\omega_0}\right) - \frac{\varepsilon_{\omega_0}\mu_{\omega_0}}{c^2} = 0, \tag{21}$$

where $\varepsilon_{\omega_0} \equiv 1+\chi_{\omega_0}$ and $\mu_{\omega_0} \equiv 1+\chi^m_{\omega_0}$. The considerations are valid for nonrelativistic velocities of the medium (i.e. first order of $\mathbf{a}$), therefore to obtain (21) the terms $\propto u^2/c^4$ have been neglected in comparison to terms $\propto u/c^2$. From (21) one easily obtains the Fresnel's drag of light effect

$$v_{ph} \equiv \frac{\omega}{k} = \frac{c}{n_{\omega_0}} + \left(1 - \frac{1}{n^2_{\omega_0}}\right)\mathbf{u}\cdot\hat{\mathbf{k}}, \tag{22}$$

where v_{ph} is phase velocity of the light and $\hat{\mathbf{k}}$ is a unit vector in direction of $\mathbf{k}$. Because the refractive index $n_{\omega_0} \equiv \sqrt{\varepsilon_{\omega_0}\mu_{\omega_0}}$ appears at frequency ω_0, it is possible to take into account additional drag effect due to optical dispersion. Using formula (A.7) one obtains

$$n_{\omega_0} = n_\omega - \frac{\partial n_\omega}{\partial\omega} n_\omega \beta\omega \tag{23}$$

(here $\beta = u/c$) and the formula (22) takes on the form

$$v_{ph} \equiv \frac{\omega}{k} = \frac{c}{n_\omega} + \left(1 - \frac{1}{n^2_\omega}\right)\mathbf{u}\cdot\hat{\mathbf{k}} - \frac{\partial n_\omega}{\partial\omega}\,\frac{\omega}{n_\omega}\mathbf{u}\cdot\hat{\mathbf{k}}. \tag{24}$$

The result well known from literature, see e.g. [8] and the references within.

4 "Optical Fields" and "Nonrelativistic Limit"

Multiplying (14) from left by the operator

$$E - e_{opt}\varphi + \vec{\alpha}\cdot(c\mathbf{P} - e_{opt}\mathbf{a}) + mc^2\beta \tag{25}$$

one obtains

$$\begin{aligned}\big[(E - e_{opt}\varphi)^2 - (c\mathbf{P} - e_{opt}\mathbf{a})^2 - m^2c^4 \\ + \hbar c e_{opt}\mathbf{\Sigma}\cdot\mathbf{B}_{opt} - i e_{opt}\hbar c\vec{\alpha}\cdot\mathbf{E}_{opt}\big]\psi_\omega = 0,\end{aligned} \tag{26}$$

$$\mathbf{\Sigma} = \begin{bmatrix}\mathbf{S} & 0\\ 0 & \mathbf{S}\end{bmatrix} \tag{27}$$

is the six-component spin operator of the photon.

$$\mathbf{B}_{opt} = \nabla\times\mathbf{a} \tag{28}$$

is an "optical magnetic field"—some analog of the magnetic field.

$$\mathbf{E}_{opt} = -\frac{1}{c}\frac{\partial\mathbf{a}}{\partial t} \tag{29}$$

is an “optical electric field”—a formal analog of the electric field. The definition (29) is valid only in a homogeneous medium. In the case of inhomogeneity the additional terms appear $-\nabla\varphi$ and as well the derivative of photon mass. The time derivative appears in (29) because of the substitution

$$E \to i\hbar\partial_t. \tag{30}$$

It is valid for a photon in a stationary state. Strictly, in such a state the vector potential should not depend on time. Therefore I assume here that the flow $\mathbf{u}$ and the susceptibilities do not change significantly on a scale of an optical wave length and in one optical cycle. Note that for a stationary flow of a homogeneous medium the “electric optical field” always disappears. To write equation (26) the following equality is needed

$$(\vec{\alpha}\cdot\mathbf{p})(\vec{\alpha}\cdot\mathbf{p}) = (\vec{\alpha}\cdot\mathbf{p})^2 = \mathbf{p}^2 + i\boldsymbol{\Sigma}\cdot(\mathbf{p}\times\mathbf{p}), \tag{31}$$

it is valid on condition that $\mathbf{p}\cdot\psi_\omega = 0$. It will be assumed that this divergence condition is at least approximately valid.

In the electron Dirac theory it is a standard procedure to obtain the “nonrelativistic limit” of the Dirac equation. (Note that now the word “nonrelativistic” has been used in different meaning as previously. Do not confuse the “nonrelativistic limit” with “nonrelativistic velocities of the medium”.) To this end the total energy of the electron is usually written down in the following way

$$E = mc^2 + E_r \tag{32}$$

and next the following approximation is made

$$E_r - e\varphi \ll 2mc^2. \tag{33}$$

In the case of photon, the validity of the procedure is not so obvious. The electric charge e should be replaced by optical charge e_{opt} and electromagnetic scalar potential by potential of the medium defined by (12). Consider for example the case of nonmagnetic media ($\Gamma_\omega = 0$). Using the given definitions of the photon mass, potential of the medium and optical charge the inequality (33) can be expressed in the form

$$1 \ll \chi. \tag{34}$$

Thus, the “nonrelativistic limit” of the photon Dirac equation means simply a large value of the dielectric susceptibility (or equivalently large value of refractive index). In this limit (26) becomes

$$E_r\psi_\omega = \left[\frac{1}{2m}\left(\mathbf{P} - \frac{e_{opt}}{c}\mathbf{a}\right)^2 + e_{opt}\varphi - \frac{e_{opt}\hbar}{2mc}\boldsymbol{\Sigma}\cdot\mathbf{B}_{opt} + \frac{ie_{opt}\hbar}{2mc}\vec{\alpha}\cdot\mathbf{E}_{opt}\right]\psi_\omega. \tag{35}$$

In a way it is an optical version of the Pauli equation.

5 Mechanical Faraday Effect as a Precession of Photon Spin

It is a known effect that mechanical rotation of a material sample can induce a rotation of the plane of linear polarization of a light beam traversing the sample along the rotation axis [9–11]. It will be shown that this effect can be described as a precession of a photon spin in the "optical magnetic field". Assume the rotation is rigid and the medium is homogeneous and nonmagnetic. Consider first what a result can be obtained from (35). Homogeneity and stationarity of the motion mean that the $\mathbf{E}_{opt}$ disappears. For a constant $\mathbf{B}_{opt}$ the first term in (35) can be written as

$$\left(\mathbf{P} - \frac{e_{opt}}{c}\mathbf{a}\right)^2 = \mathbf{P}^2 - \frac{e_{opt}}{c}\mathbf{B}_{opt} \cdot \mathbf{L}, \tag{36}$$

where $\mathbf{L}$ is orbital angular momentum. The terms proportional to $\mathbf{a}^2$ do not enter this equation because only nonrelativistic velocities of the medium are concerned. Thus, for a state of photon with a well defined momentum along the rotation axis the vectors $\mathbf{B}_{opt}$ and $\mathbf{L}$ are orthogonal and (35) takes on the form

$$\hbar\omega\psi_\omega = \left[\frac{1}{2m}\mathbf{P}^2 - \frac{e_{opt}\hbar}{2mc}\boldsymbol{\Sigma} \cdot \mathbf{B}_{opt}\right]\psi_\omega. \tag{37}$$

Note that the equality $E_r - e_{opt}\varphi = \hbar\omega$ has been used, valid for a nonmagnetic medium. After some manipulation one obtains from (37) two independent equations of identical form for $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$. They contain the same information, therefore it is enough to consider only one of them, e.g. for $\mathbf{E}_\omega$

$$\hbar\omega\mathbf{E}_\omega = \left[\frac{1}{2m}\mathbf{P}^2 - \frac{e_{opt}\hbar}{2mc}\mathbf{S} \cdot \mathbf{B}_{opt}\right]\mathbf{E}_\omega. \tag{38}$$

Assume the field $\mathbf{B}_{opt}$ in the z-direction. Two eigenstates and eigenvalues of the operator S_z are

$$S_z\hat{\mathbf{e}}_\pm = \pm\hat{\mathbf{e}}_\pm, \tag{39}$$

$$\lambda = 1, \qquad \hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}; \qquad \lambda = -1, \qquad \hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}. \tag{40}$$

The third eigenvalue $\lambda = 0$ is excluded, because of the transversality of the electromagnetic wave. In general, the state of the photon propagating in the z-direction can be described in the following way [12]

$$\hat{\mathbf{e}} = \cos\alpha\hat{\mathbf{e}}_x + e^{i\delta}\sin\alpha\hat{\mathbf{e}}_y \equiv \begin{pmatrix}\cos\alpha\\ e^{i\delta}\sin\alpha\\ 0\end{pmatrix}. \tag{41}$$

Assuming that for $z = 0$ the photon is polarized in the x-direction, we can write

$$\mathbf{E}_\omega = E_\omega\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}e^{ikz}. \tag{42}$$

For $z > 0$ the state of the photon will change. However, in every moment it can be expressed as a linear combination of the vectors $\hat{\mathbf{e}}_\pm$

$$\mathbf{E}_\omega = \frac{E_\omega}{\sqrt{2}}\left(\hat{\mathbf{e}}_+e^{ik_+z} + \hat{\mathbf{e}}_-e^{ik_-z}\right). \tag{43}$$

Putting (43) into (38) one obtains equations for the wave vectors k_+ and k_-

$$\hbar\omega = \frac{\hbar^2 k_+^2}{2m} - \mu_{opt} B_{opt}, \qquad \hbar\omega = \frac{\hbar^2 k_-^2}{2m} + \mu_{opt} B_{opt}, \tag{44}$$

where "optical magneton"

$$\mu_{opt} = \frac{e_{opt}\hbar}{2mc} = \frac{\hbar c}{\chi}. \tag{45}$$

Because the plane of rotation of the medium is perpendicular to the direction of the propagating light, and for a nonmagnetic medium $\mathbf{a} = -\chi \frac{\mathbf{u}}{c}$, thus from $\mathbf{B}_{opt} = \nabla \times \mathbf{a}$ we have

$$\mu_{opt} B_{opt} = -\hbar |\nabla \times \mathbf{u}| = -2\hbar\omega_m. \tag{46}$$

Usually the angular velocity of the medium ω_m is much smaller than the frequency of light ω. Therefore,

$$k_+ \approx \sqrt{\chi}\frac{\omega}{c}\left(1 - \frac{\omega_m}{\omega}\right), \qquad k_- \approx \sqrt{\chi}\frac{\omega}{c}\left(1 + \frac{\omega_m}{\omega}\right). \tag{47}$$

The vector $\mathbf{E}_\omega$ (43) takes on the form

$$\begin{aligned} \mathbf{E}_\omega &= \frac{E_\omega}{\sqrt{2}} \left(\hat{\mathbf{e}}_+ e^{-i\sqrt{\chi}\frac{\omega_m}{c}z} + \hat{\mathbf{e}}_- e^{i\sqrt{\chi}\frac{\omega_m}{c}z}\right) e^{i\sqrt{\chi}\frac{\omega}{c}z} \\ &= \frac{E_\omega}{2}\left[\begin{pmatrix}1\\ i\\ 0\end{pmatrix} e^{-i\sqrt{\chi}\frac{\omega_m}{c}z} + \begin{pmatrix}1\\ -i\\ 0\end{pmatrix} e^{i\sqrt{\chi}\frac{\omega_m}{c}z}\right] e^{i\sqrt{\chi}\frac{\omega}{c}z} \\ &= E_\omega \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} e^{i\sqrt{\chi}\frac{\omega}{c}z}. \end{aligned} \tag{48}$$

It is clear now that the polarization vector precesses about the "optical magnetic field" direction. The rotation angle θ is given by

$$\theta = n\frac{\omega_m}{c}z, \tag{49}$$

where $\sqrt{\chi}$ is replaced by the refractive index because in "nonrelativistic limit" $\sqrt{\chi} = \sqrt{\varepsilon - 1} \approx \sqrt{\varepsilon} = n$.

There is no need to restrict consideration to the "nonrelativistic limit" (large refractive index) (35). One can start as well from the exact (for nonrelativistic velocities of the medium) equation (26). Doing things in the similar way as previously one may obtain from (26)

$$\left[(E - e_{opt}\varphi)^2 - c^2P^2 - m^2c^4 + \hbar c e_{opt}\mathbf{S}\cdot\mathbf{B}_{opt}\right]\mathbf{E}_\omega = 0. \tag{50}$$

It can be written in the form

$$\left[\left(n\frac{\hbar\omega}{c}\right)^2 - P^2 + \frac{\hbar\omega}{c}\hbar\mathbf{S}\cdot\mathbf{B}_{opt}\right]\mathbf{E}_\omega = 0. \tag{51}$$

This time one obtains the following equations for the wave vectors k_+, k_-

$$\left(n\frac{\hbar\omega}{c}\right)^2 - \hbar^2 k_+^2 + \frac{\hbar\omega}{c}\hbar B_{opt} = 0, \qquad \left(n\frac{\hbar\omega}{c}\right)^2 - \hbar^2 k_-^2 - \frac{\hbar\omega}{c}\hbar B_{opt} = 0. \tag{52}$$

And for small ω_m (first order of ω_m) they can be written in the form

$$k_+ \approx \frac{n\omega}{c}\left(1 - \left(1 - \frac{1}{n^2}\right)\frac{\omega_m}{\omega}\right), \qquad k_- \approx \frac{n\omega}{c}\left(1 + \left(1 - \frac{1}{n^2}\right)\frac{\omega_m}{\omega}\right). \tag{53}$$

As the final result we have

$$\mathbf{E}_\omega = E_\omega \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} e^{i\frac{n\omega}{c}z}, \tag{54}$$

where now the rotation angle θ is

$$\theta = \left(n - \frac{1}{n}\right)\frac{\omega_m}{c}z. \tag{55}$$

So far the dependence of χ on frequency has not been taken into account. Therefore the "dispersive term" predicted by Player [11] does not enter equation (55). It can be easily improved. In the rotational situation considered here the general Doppler effect relation (8) specifies to

$$\omega_0 = \omega \mp \omega_m. \tag{56}$$

It reflects the fact that in the rotating frame, in which medium is at rest, circularly polarized light with frequency ω is observed with the shifted frequency ω_0. Therefore, in the first order of ω_m the susceptibility χ_{ω_0} can be written as

$$\chi_{\omega\mp\omega_m} = \chi_\omega \mp \frac{\partial\chi_\omega}{\partial\omega}\omega_m. \tag{57}$$

The upper sign is for the wave vector k_+ and the lower for k_-. Therefore, (52) takes on the form

$$k_\pm^2 = (1 + \chi_{\omega\mp\omega_m})\frac{\omega^2}{c^2} \mp 2\chi_{\omega\mp\omega_m}\frac{\omega}{c^2}\omega_m. \tag{58}$$

And thus

$$k_\pm \approx \frac{n\omega}{c} \mp \left(n - \frac{1}{n}\right)\frac{\omega_m}{c} \mp \frac{\omega}{c}\frac{\partial n}{\partial\omega}\omega_m. \tag{59}$$

Following the same reasoning as before we obtain now for the rotation angle θ the value

$$\theta = \left(n - \frac{1}{n}\right)\frac{\omega_m}{c}z + \frac{\omega}{c}\frac{\partial n}{\partial\omega}\omega_m z. \tag{60}$$

The last term is the above-mentioned "dispersive term".

The constant "optical magnetic field" can be realized in many ways. The field $\mathbf{B}_{opt}$ is gauge invariant and there are many ways of choosing vector potential $\mathbf{a}$ of the medium. It may be the symetric gauge $\mathbf{a} = (B_{opt}/2)(-y, x, 0)$ or Landau gauge $\mathbf{a} = B_{opt}(-y, 0, 0)$ or any other giving $\hat{\mathbf{z}} \cdot (\nabla \times \mathbf{a}) = B_{opt} = const$. In contrast to electrodynamic vector potential, the vector potential of the medium is an observable quantity. It is determined by the motion of the medium. Therefore, the changes of the gauge mean physically different flows of the medium.

6 Landau Levels in Moving Medium

A nonrelativistic quantum-mechanical treatment of an electron in a constant and uniform magnetic field was considered for the first time by Landau [13] in 1930. It was shown there that energy of the electron in the field is quantized. The energy levels are called Landau levels. In this Section it will be shown that in the optical magnetic field the energy (or equivalently frequency) of photon is quantized. To refer directly to Landau approach (35) will be used to consider the circularly polarized states of light moving in the uniform constant optical magnetic field $(0, 0, B_{opt})$. Assuming homogeneity of the medium and stationarity of its motion the optical electric field in (35) vanishes and the constant scalar potential can be included into the energy eigenvalues. Therefore, (35) takes on the form

$$\hbar\omega\psi_\omega = \left[\frac{1}{2m}(-i\hbar\nabla - e_{opt}\mathbf{a}/c)^2 - \frac{e_{opt}\hbar}{2mc}\Sigma_z B_{opt}\right]\psi_\omega. \tag{61}$$

The problem is equivalent to the harmonic oscillator problem and therefore the eigenstates and eigenvalues are well known, see e.g. [13]. The discrete eigenvalues (Landau levels) are given in the following form

$$\hbar\omega = \frac{\hbar k_z^2}{2m} + \left(n + \frac{1}{2} \pm \frac{1}{2}\lambda\right)\hbar\omega_{B_{opt}}, \tag{62}$$

$$\text{where } \omega_{B_{opt}} = \frac{e_{opt}B_{opt}}{mc}, \tag{63}$$

integers $n = 0, 1, 2, \dots,$ the eigenvalues of the operator S_z, $\lambda = -1, 1$ ($\lambda = 0$ is excluded because of transversality of the electromagnetic wave). Now one may use the definitions of the photon mass m, optical charge e_{opt}, and optical magnetic field B_{opt} to obtain the dispersion relation of the photon in the medium

$$\omega = \frac{ck_z}{\sqrt{\chi_{\omega_0}}} + (2n + 1 \pm \lambda)\,|\nabla \times \mathbf{u}|. \tag{64}$$

To write the relation it was supposed that $|\frac{ck_z}{\sqrt{\chi_{\omega_0}}}| \gg |(2n + 1 \pm \lambda)\nabla \times \mathbf{u}|$. If the optical magnetic field is realized by the rigid rotation of the medium with the angular velocity of the medium ω_m, then the dispersion relation (64) may be rewritten in the form

$$\omega = \frac{ck_z}{\sqrt{\chi_{\omega_0}}} + 2\omega_m\,(2n + 1 \pm \lambda)\,. \tag{65}$$

For a given wave vector component k_z the frequency of the propagating light takes the discrete values. Note that the states of light are given rather by Hermite polynomials than simply by plane waves [13].

There is no need to restrict considerations to large values of χ, because solutions of the exact Dirac equation for the electron in a constant magnetic field are already well known, see e.g. [14]. Therefore one may use the exact optical Dirac equation (14) for obtaining the following eigenvalue equation

$$(E - e_{opt}\varphi)^2 = (c\hbar k_z)^2 + m^2c^4 + c\hbar e_{opt}B_{opt}(2n + 1 - \lambda), \quad n = 0, 1, 2, \ldots. \tag{66}$$

Using definitions of the quantities E, m, e_{opt}, B_{opt} one obtains instead of (64)

$$\omega = \frac{ck_z}{n_{\omega 0}} + \left(1 - \frac{1}{n^2_{\omega 0}}\right)|\nabla \times \mathbf{u}|(2n + 1 - \lambda). \tag{67}$$

The result differs from (64) only in the term $1/n^2_{\omega 0}$. In (65) and (67) $\chi_{\omega 0}$ and $n_{\omega 0}$ appear at frequency ω_0. One may easily improve the equations taking into account a "dispersive term" in the similar way as it has been done for Fresnel's drag and mechanical Faraday effect. The term comes from $ck_z/n_{\omega 0}$ only, because the second term on the right hand side in (67) gives second order of $\mathbf{u}$ corrections which may be neglected.

7 Summary

It has been shown that the formal analogy in description of the photon in moving media and electron in electromagnetic field can be useful to describe and sometimes to predict new phenomena. In the terms of the analogy the well known Fresnel's drag of light and mechanical Faraday effect have been obtained. Additionally the approach has revealed that in the rotating medium apart of the mechanical Faraday effect there is also some quantization of the frequency of light. Up to my knowledge the appearance of the Landau frequencies in the rotating medium is yet unknown result.

Appendix

The susceptibility dispersion caused by motion of the medium was not taken into account in [5]. The nondispersive Minkowski relations for a moving medium can be found in [6]. The derivation of them clearly suggests that in the case of dispersive medium the relations must be modified. Below I present my proposition of generalization of the Minkowski relations for nonrelativistic velocities of the medium

$$\begin{aligned} \mathbf{D}_\omega &= \varepsilon_{\omega 0}\mathbf{E}_\omega - (1 - \varepsilon_{\omega 0}\mu_{\omega 0})\vec{\beta} \times \mathbf{H}_\omega, \\ \mathbf{B}_\omega &= \mu_{\omega 0}\mathbf{H}_\omega + (1 - \varepsilon_{\omega 0}\mu_{\omega 0})\vec{\beta} \times \mathbf{E}_\omega, \end{aligned} \tag{A.1}$$

and for relativistic velocities of the medium

$$\begin{aligned} \mathbf{D}_\omega &= \varepsilon_{\omega 0}\mathbf{E}_{\omega\parallel} + \frac{1 - \beta^2}{1 - \varepsilon_{\omega 0}\mu_{\omega 0}\beta^2}\varepsilon_{\omega 0}\mathbf{E}_{\omega\perp} - \frac{1 - \varepsilon_{\omega 0}\mu_{\omega 0}}{1 - \varepsilon_{\omega 0}\mu_{\omega 0}\beta^2}\vec{\beta} \times \mathbf{H}_{\omega\perp}, \\ \mathbf{B}_\omega &= \mu_{\omega 0}\mathbf{H}_{\omega\parallel} + \frac{1 - \beta^2}{1 - \varepsilon_{\omega 0}\mu_{\omega 0}\beta^2}\mu_{\omega 0}\mathbf{H}_{\omega\perp} - \frac{1 - \varepsilon_{\omega 0}\mu_{\omega 0}}{1 - \varepsilon_{\omega 0}\mu_{\omega 0}\beta^2}\vec{\beta} \times \mathbf{E}_{\omega\perp}. \end{aligned} \tag{A.2}$$

These relations may be compared with (32) and (37) in [5]. The meaning of the frequencies ω and ω_0 has been explained in the present paper in Sect. 2. Only for a medium at rest $\omega = \omega_0$.

From (3) and (A.1) for nonrelativistic velocities they are simplified to

$$1+\frac{\Omega_\omega}{\hbar\omega}=\varepsilon_{\omega 0}\equiv 1+\chi_{\omega 0},\qquad 1+\frac{\Gamma_\omega}{\hbar\omega}=\mu_{\omega 0}\equiv 1+\chi^m_{\omega 0},\qquad \mathbf{a}=(1-\varepsilon_{\omega 0}\mu_{\omega 0})\vec{\beta},\tag{A.3}$$

I have used the result in this paper, e.g. (7).

However, in a medium moving with relativistic velocity (though the light source and the observer are at relative rest) the observed frequency ω_{obs} is different from the frequency ω emitted by the source. The frequency shift is given by the following formula

$$\omega_{obs}=\frac{1-\beta^2}{1-\varepsilon_{\omega 0}\mu_{\omega 0}\beta^2}\,\omega.\tag{A.4}$$

It can be proved elementary if one considers first the Doppler shift between the source and the medium

$$\omega_0=\frac{\sqrt{1-\beta^2}}{1+n\beta}\,\omega,\tag{A.5}$$

and then treats the medium as a new source for the observer. The second step produces the next Doppler shift

$$\omega_{obs}=\frac{\sqrt{1-\beta^2}}{1-n\beta}\,\omega_0,\tag{A.6}$$

where $n=\sqrt{\varepsilon_{\omega 0}\mu_{\omega 0}}$. The formula (A.4) contains the both contributions.

Note also that for nonrelativistic velocity of the medium the following approximations result from (A.4) and (A.5)

$$\omega_{obs}\approx\omega,\qquad \omega_0\approx(1-n\beta)\omega.\tag{A.7}$$

These approximations were used in Sect. 5.

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